L4.1

1, 2, ... n qubits

Pure State

$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

$\langle\psi|\psi\rangle = 1$, global phase

$4 \in \mathbb{C} \Rightarrow 8 \in \mathbb{R} \quad - 2$

6 parameters

⚠ more than 2 Bloch spheres

n-qubits $\quad 2 \cdot 2^n - 2 = 2^{n+1} - 2$

Mixed States

$\rho = |\psi\rangle\langle\psi|$

$$\rho = \begin{pmatrix} \rho_{00} & \dots & \rho_{03} \\ \vdots & & \\ \rho_{30} & & \rho_{33} \end{pmatrix}$$

$\rho^\dagger = \rho \qquad tr(\rho) = 1$

n-qubits $2^n \cdot 2^n - 1 = 2^{2n} - 1$ L4.2

$s_x^1 = tr(\rho \sigma_x^1)$, $s_y^1 = \dots$, s_z^1

s_x^2, s_y^2, s_z^2

$s_{xx} = tr(\rho \sigma_x^1 \sigma_x^2)$,

$s_{xy} = \dots$

$s_{xz} = \dots$

$s_{yx} = \dots$

$\vdots$

	pure	mixed
1	2	3
2	6	15
3	14	63
4	30	255
n	$2^{n+1}-2$	$2^{2n}-1$

L4.3 Quantum Gates

- Unitary Evolution

U $\quad 2^n \times 2^n$ matrix

$U^\dagger U = \mathbb{1} \qquad U^\dagger = U^{-1}$

$U = e^{-iHt/\hbar}$

single qubit gates

$-\boxed{X}- \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$-\boxed{Y}- \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$-\boxed{Z}- \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$-\boxed{H}- \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$-\boxed{S}- \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

arbitrary angle rotations

$$R_x(\theta) = e^{-i\theta\sigma_x/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_x$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta\sigma_y/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_y$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta\sigma_z/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_z$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix}$$

phase gate S is a rotation by $\frac{\pi}{2}$
around the z-axes and
a phas $e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

L4.5 <u>theorem</u>

if U is an arbitrary single qubit operation, i.e. unitary 2x2 matrix
there exists an angle θ, an axis $\hat{n}$ and a phas $e^{i\alpha}$, such that

$$U = e^{i\alpha} R_{\hat{n}}(\theta)$$

with $\alpha, \theta \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 = 1$
$n_x, n_y, n_z \in \mathbb{R}$

in words: every single qubit operation is a rotation

$$R_{\hat{n}}(\theta) = e^{-i\theta \hat{n}\vec{\sigma}/2}$$

$$= \cos\left(\frac{\theta}{2}\right) \mathbb{1} - i \sin\left(\frac{\theta}{2}\right)\left(n_x \sigma_x + n_y \sigma_y + n_z \sigma_z\right)$$

Example: Hadamard

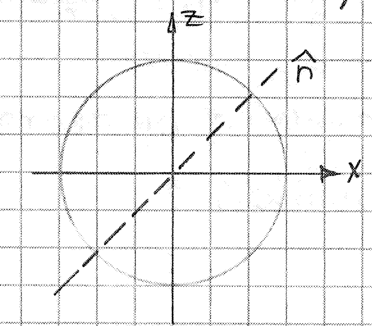
$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

H changes from z to x basis

$$H^2 = \mathbb{1}$$

$$H = e^{i\frac{\pi}{2}} R_{\hat{n}}(\pi) \qquad \hat{n} = \frac{1}{\sqrt{2}}(1,0,1)$$



X-Y decomposition

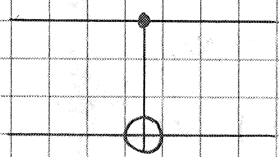
$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)$$

every qubit operation can be performed with rotations around the x and y axes

L4.7

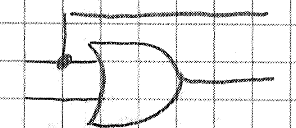
two qubit gates

controlled not



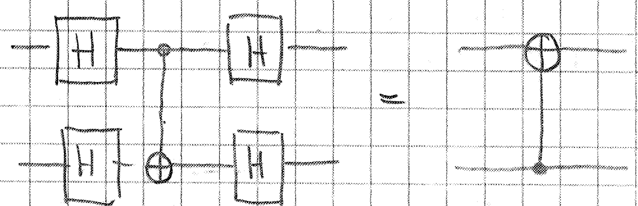
$$CNOT = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

classical equivalent



XOR

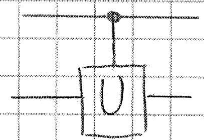
warning: do not think "classical"



wrongly one could think that the first qubit just is affected by two Hadamard gates, i.e. the identity theorem

single qubit rotations and CNOT are universal, i.e. every unitary operation can efficiently be approximated with $R_{\hat{n}}(\theta)$ and CNOT

other two qubit gates



$$\text{controlled } U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & U_{00} & U_{01} \\ & & U_{10} & U_{11} \end{pmatrix}$$

$$\text{SWAP} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

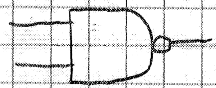
$$\sqrt{\text{SWAP}} = \begin{pmatrix} 1 & & & \\ & \frac{1+i}{2} & \frac{1-i}{2} & \\ & \frac{1-i}{2} & \frac{1+i}{2} & \\ & & & 1 \end{pmatrix}$$

$\sqrt{\text{SWAP}}$ is universal

$\sqrt{\text{SWAP}}$ is universal

Comparison with Classical Logic

Boolean logic



NAND

0	0	1
0	1	1
1	0	1
1	1	0

NAND is universal for Boolean logic
(single gate is enough)

note: for quantum gates
the x-axes is always the
time axes